Journal of Computational and Applied Mathematics 388 (2021) 113313

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Application of optimal quadrature formulas for reconstruction of CT images

Abdullo R. Hayotov [a,b,*], Soomin Jeon [a], Kholmat M. Shadimetov [b,c]

[a] Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 34141, Republic of Korea
[b] V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 81, M.Ulugbek str., Tashkent 100170, Uzbekistan
[c] Tashkent Railway Engineering Institute, 1 Odilxojaev str., Tashkent 100167, Uzbekistan

ARTICLE INFO

Article history:
Received 24 April 2020
Received in revised form 4 October 2020

MSC:
41A05
41A15
65D32

Keywords:
Optimal quadrature formula
Sobolev space
The error functional
Fourier transform
Radon transform
The filtered back-projection method
CT image reconstruction

ABSTRACT

In the present paper, the construction process of the optimal quadrature formulas for weighted integrals is presented in the Sobolev space $\tilde{L}_2^{(m)}(0,1]$ of complex-valued periodic functions which are square integrable with mth order derivative. In particular, optimal quadrature formulas are given for Fourier coefficients. Here, using these optimal quadrature formulas the approximation formulas for Fourier integrals $\int_a^b e^{2\pi i\omega x}f(x)dx$ with $\omega \in \mathbb{R}$ are obtained. In the cases $m = 1,\ 2$ and 3, the obtained approximation formulas are applied for reconstruction of Computed Tomography (CT) images coming from the filtered back-projection method. Compared with the optimal quadrature formulas in non-periodic case, the approximation formula for the periodic case is much simpler, therefore it is easy to implement and costs less computation.

1. Introduction

The Fourier transforms are widely used in science and technology, particularly, in the problems of Computed Tomography (CT). It is well-known that when complete continuous X-ray data are available, CT images can be reconstructed exactly using the filtered back-projection formula (see, for instance, [1–3]). This formula consequentially uses the Radon transform, the Fourier transforms and the back-projection formula. Fourier transforms play an important role in the filtered back-projection method of CT image reconstruction. Since in practice we have finite discrete values of the Radon transform, we have to approximately calculate the Fourier transforms in the filtered back-projection method of CT. Therefore, one has to consider the problem of approximate calculation of the integral

$$I(f,\omega) = \int_a^b e^{2\pi i\omega x} f(x)dx \tag{1.1}$$

with $\omega \in \mathbb{R}$. Such type of integrals are called *the integrals with highly or strongly oscillating integrands*. In most cases it is impossible to get the exact values of such integrals. They are mainly computed by special effective numerical methods. One of the first numerical integration formula for the integral (1.1) was obtained by Filon [4] in 1928, using a combination

* Corresponding author at: V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 81, M.Ulugbek str., Tashkent 100170, Uzbekistan.
E-mail addresses: hayotov@mail.ru (A.R. Hayotov), soominjeon@kaist.ac.kr (S. Jeon), kholmatshadimetov@mail.ru (K.M. Shadimetov).

https://doi.org/10.1016/j.cam.2020.113313

of second degree polynomials. Then for integrals of different types of highly oscillating integrands many special effective methods have been developed (see, for example, [5–13], for more review see [14–16] and references therein).

We consider the Sobolev space $L_2^{(m)}[a,b]$ of complex-valued functions f defined on the interval $[a,b]$, which possess an absolute continuous $(m-1)$-st derivative on $[a,b]$ and whose mth order derivative is square integrable [17,18]. The space $L_2^{(m)}[a,b]$ under the pseudo-inner product

$$\langle f,g\rangle_m=\int_a^b f^{(m)}(x)\,\bar{g}^{(m)}(x)\mathrm{d}x, \tag{1.2}$$

is a Hilbert space if we identify functions that differ by a polynomial of degree $(m-1)$ (see, for example, page 213 of [19]). Here, in (1.2), $f^{(m)}$ is the mth order derivative of the function f and $\bar{g}$ is the complex conjugate to the function g. We note that every element of the space $L_2^{(m)}[a,b]$ is a class of functions which differ from each other by a polynomial of degree $(m-1)$. The semi-norm of the function f is correspondingly defined by the formula

$$\|f\|_{L_2^{(m)}[a,b]}=\langle f,f\rangle_m^{1/2}.$$

Furthermore, we deal with the corresponding Sobolev space $\tilde{L}_2^{(m)}(a,b]$ of periodic complex-valued functions from the space $L_2^{(m)}[a,b]$ with period $T=b-a$. Notice that every element of the space $\tilde{L}_2^{(m)}(a,b]$ satisfies the following condition of T-periodicity

$$f(x+T\beta)=f(x),\ x\in\mathbb{R},\ \beta\in\mathbb{Z}$$

and is a class of functions f which differ from each other by a constant term.

It should be noted that in [20,21] and [22], based on Sobolev's method, the problem of construction of optimal quadrature formulas in the sense of Sard for numerical calculation of integral (1.1) with integer ω was solved in Hilbert spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$, respectively. Recently, in the works [23,24], optimal quadrature formulas for approximation of Fourier integrals (1.1) with real ω were constructed in the Sobolev space $L_2^{(m)}[a,b]$. The obtained optimal quadrature formulas were applied to reconstruct the X-ray Computed Tomography image by approximating Fourier transforms.

We notice that optimal quadrature formulas for approximate integration of the Fourier coefficients (1.1) with integer ω in the Sobolev space $\tilde{L}_2^{(m)}$ of periodic functions were obtained by Shadimetov [25,26].

The aim of the present work is to get approximation formulas for numerical calculation of the integral (1.1) with real ω using optimal quadrature formulas for Fourier coefficients constructed in [25, Theorem 1] and [26, Corollary 2] in the Sobolev space $\tilde{L}_2^{(m)}(0,1]$ of complex-valued periodic functions, then to apply the approximation formulas for numerical reconstruction of CT images employing the filtered back-projection method.

The rest of the paper is organized as follows. In Section 2 the construction process of the optimal quadrature formulas for weighted integrals is presented in the Sobolev space $\tilde{L}_2^{(m)}(0,1]$. In particular, for Fourier coefficients $\int_0^1 \mathrm{e}^{2\pi\mathrm{i}\omega x}f(x)\mathrm{d}x$ with $\omega\in\mathbb{Z}$, the optimal quadrature formulas are given. In Section 3, Using these optimal quadrature formulas the approximation formulas for numerical calculation of the Fourier integrals (1.1) with real ω are obtained. In Section 4, the obtained approximation formulas for the cases $m=1,\ 2$ and 3 are applied to reconstruct CT images by approximating Fourier transforms in the filtered back-projection method.

2. Optimal quadrature formulas for weighted integrals in the Sobolev space of periodic functions

For functions f of the space $\tilde{L}_2^{(m)}(0,1]$ we consider a quadrature formula for the weighted integrals with a weight function $p(x)$ in the following form

$$\int_0^1 p(x)f(x)\mathrm{d}x\cong\sum_{k=1}^{N}C_kf(hk), \tag{2.1}$$

where $p(x)=v(x)+\mathrm{i}w(x)$ is a 1-periodic integrable function, parameters $C_k=V_k+\mathrm{i}W_k$, $k=1,2,\ldots,N$, are coefficients of the quadrature formula, $h=1/N$ and $N\in\mathbb{N}$.

The difference

$$\begin{aligned}(\ell,f)&=\int_0^1 p(x)f(x)\mathrm{d}x-\sum_{k=1}^{N}C_kf(hk)\\&=\int_0^1 \ell(x)f(x)\mathrm{d}x\end{aligned} \tag{2.2}$$

is called *the error* of the quadrature formula (2.1) and gives the value of the error functional ℓ at the function f. The error functional ℓ has the following representation:

$$\ell(x)=p(x)-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-\beta-hk), \tag{2.3}$$

where δ is the Dirac delta-function.

Since the error functional ℓ is defined on the space $\tilde{L}_2^{(m)}(0,1]$, due to the fact that any periodic algebraic polynomial is a constant [17, p. 683] the following condition should be imposed

$$(\ell, 1) = 0. \tag{2.4}$$

This condition does not depend on m. This equality means exactness of the quadrature formula (2.1) for constant terms and it is equivalent to the following equality:

$$\sum_{k=1}^{N} C_k = \int_0^1 p(x)\mathrm{d}x. \tag{2.5}$$

It is clear that the coefficients C_k are variable parameters of the quadrature formula (2.1). A quadrature formula of the form (2.1) which has the error functional with the minimum norm by coefficients C_k for a given number N of the nodes is called *the optimal quadrature formula* in the space $\tilde{L}_2^{(m)}(0,1]$.

We note that in this section for completeness we give construction of the optimal quadrature formula (2.1) in the space $\tilde{L}_2^{(m)}(0,1]$ of 1-periodic complex-valued functions based on the results of the work [26]. Further, we also use a concept of functions of a discrete argument and the corresponding operations (see [17] and [18]).

The function $f(hk)$ is *a function of discrete argument* if it is given on some set of integer values of k, where h is a small positive parameter. *The convolution* of two discrete argument functions $f(hk)$ and $g(hk)$ is defined as

$$f(hk) * g(hk) = \sum_{l=-\infty}^{\infty} f(hl)g(hk - hl).$$

Here we use *the discrete analogue* $D_m(hk)$ of the differential operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}$, which has the following form [27]:

$$D_m(hk) = \frac{(2m-1)!}{h^{2m}} \begin{cases} \sum\limits_{n=1}^{m-1} A_n q_n^{|k|-1} & \text{for } |k| \geq 2, \\ 1 + \sum\limits_{n=1}^{m-1} A_n & \text{for } |k| = 1, \\ -2^{2m-1} + \sum\limits_{n=1}^{m-1} \frac{A_n}{q_n} & \text{for } k = 0, \end{cases} \tag{2.6}$$

where $A_n = \frac{(1-q_n)^{2m+1}}{E_{2m-1}(q_n)}$, $E_{2m-1}(x)$ is the Euler–Frobenius polynomial of degree $(2m-1)$, q_n are the roots of the Euler–Frobenius polynomial $E_{2m-2}(x)$ with $|q_n| < 1$ and h is a small positive parameter.

The Euler–Frobenius polynomials $E_k(x) = \sum_{s=0}^{k} \mathrm{a}_s x^s$ of degree k are defined as (see [28])

$$E_k(x) = \frac{(1-x)^{k+2}}{x} \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^k \frac{x}{(1-x)^2}, \quad k = 0, 1, 2, \ldots$$

and the coefficients of this polynomial are expressed by the following formula which was obtained by Euler:

$$\mathrm{a}_s = \sum_{j=0}^{s} (-1)^j \binom{k+2}{j} (s+1-j)^{k+1}, \quad s = 0, 1, \ldots, k.$$

We remind the following properties of the operator $D_m(hk)$ from [17,26,27]:

$$hD_m(hk) * (hk)^{\alpha} = 0, \quad \alpha < 2m, \tag{2.7}$$

$$hD_m(hk) * B_{2m}(hk) = \Phi(hk) - h, \tag{2.8}$$

where

$$B_{2m}(x) = \sum_{\beta \neq 0} \frac{\mathrm{e}^{-2\pi \mathrm{i} \beta x}}{(2\pi \mathrm{i} \beta)^{2m}} \tag{2.9}$$

is the Bernoulli polynomial of degree $2m$, $\Phi(hk) = \sum_{\gamma=-\infty}^{\infty} \delta_{\mathrm{d}}(hk - \gamma)$ and

$$\delta_{\mathrm{d}}(hk - \gamma) = \begin{cases} 1, & hk - \gamma = 0, \\ 0, & hk - \gamma \neq 0. \end{cases}$$

Also for a discrete argument function $g(hk)$ the following equality is known (see, for instance, [26])

$$g(hk) = \big(g(hk)\chi_{(0,1]}(hk)\big) * \Phi(hk), \tag{2.10}$$

where $\chi_{(0,1]}(hk)$ is the discrete argument function corresponding to the characteristic function $\chi_{(0,1]}(x)$ of the interval $(0, 1]$.

Next, in this section, for construction of the optimal quadrature formula (2.1) in the space $\tilde{L}_2^{(m)}(0,1]$, first the extremal function of the error functional ℓ is found, then using this extremal function the norm of the error functional is calculated, next minimizing this norm by coefficients the system of linear equations is obtained, and uniqueness of the solution for this system is discussed. Finally, using properties of the discrete analogue $D_m(hk)$ of the differential operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}$, the following results are obtained.

Theorem 1. *Among all quadrature formulas of the form (2.1) with the error functional (2.3) in the space $\tilde{L}_2^{(m)}(0,1]$ of 1-periodic complex-valued functions there exists a unique optimal quadrature formula with coefficients having the following representation:*

$$\mathring{C}_k = h\int_0^1 p(x)\left(1 + D_m(hk) * B_{2m}(x-hk)\right)\mathrm{d}x, \quad k = 1,2,\ldots,N, \tag{2.11}$$

where $D_m(hk)$ is the discrete analogue of the differential operator $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}$, $B_{2m}(x)$ is the Bernoulli polynomial of degree $2m$ defined by (2.9).

Corollary 1. *For $p(x) = \mathrm{e}^{2\pi \mathrm{i}\omega x}$ with $\omega \in \mathbb{Z}\backslash\{0\}$ the coefficients of optimal quadrature formulas (2.1) with the error functional (2.3) have the form*

$$\mathring{C}_k = h\left(\frac{\sin \pi\omega h}{\pi\omega h}\right)^{2m} \frac{(2m-1)!\mathrm{e}^{2\pi \mathrm{i}\omega hk}}{2\sum_{l=0}^{m-2} \mathrm{a}_l \cos[2\pi\omega(m-1-l)] + \mathrm{a}_{m-1}}, \tag{2.12}$$

where

$$\mathrm{a}_l = \sum_{j=0}^{l} (-1)^j \binom{2m}{j} (l+1-j)^{2m-1}$$

is the coefficients of the Euler–Frobenius polynomial $E_{2m-2}(x)$ of degree $(2m-2)$.

Corollary 2. *Theorem 1 implies that for $p(x) = 1$ the rectangular quadrature formula is the optimal one in the space $\tilde{L}_2^{(m)}(0,1]$ of 1-periodic complex-valued functions as in the space of 1-periodic real-valued functions [17] and the coefficient have the form*

$$\mathring{C}_k = h, \ \ k = 1,2,\ldots,N. \tag{2.13}$$

Now we start construction of the optimal quadrature formula (2.1) in the space $\tilde{L}_2^{(m)}(0,1]$ of 1-periodic complex-valued functions.

2.1. An extremal function and the norm of the error functional for the quadrature formula (2.1)

In order to find the explicit form for the norm of the error functional ℓ in the space $\tilde{L}_2^{(m)}(0,1]$ we use its extremal function. The element u of the space $\tilde{L}_2^{(m)}(0,1]$ is said to be *the extremal function* for the given functional ℓ if the following equality holds [17]:

$$(\ell, u) = \|\ell\|_{\tilde{L}_2^{(m)*}(0,1]} \|u\|_{\tilde{L}_2^{(m)}(0,1]},$$

where $\tilde{L}_2^{(m)*}$ is the conjugate space to $\tilde{L}_2^{(m)}$.

Since $\tilde{L}_2^{(m)}(0,1]$ is a Hilbert space with the inner product (1.2) then by the Riesz theorem on general form of a linear continuous functional on Hilbert spaces, for the error functional $\ell \in \tilde{L}_2^{(m)*}(0,1]$ there exists a unique function $\psi_\ell \in \tilde{L}_2^{(m)}(0,1]$ (up to constant terms) such that for any $f \in \tilde{L}_2^{(m)}(0,1]$ the following equality is fulfilled

$$(\ell, f) = \langle \psi_\ell, f \rangle_m. \tag{2.14}$$

Moreover, the equality $\|\ell\|_{\tilde{L}_2^{(m)*}(0,1]} = \|\psi_\ell\|_{\tilde{L}_2^{(m)}(0,1]}$ holds. In particular, from (2.14) when $f = \psi_\ell$, we have

$$(\ell, \psi_\ell) = \langle \psi_\ell, \psi_\ell \rangle_m = \|\psi_\ell\|^2_{\tilde{L}_2^{(m)}(0,1]} = \|\ell\|^2_{\tilde{L}_2^{(m)*}(0,1]}.$$

It is clear that the solution of Eq. (2.14) is the extremal function ψ_ℓ. Thus, in order to get the norm of the error functional we should find the extremal function ψ_ℓ from Eq. (2.14). We calculate the square of the norm of the error functional as

$$\|\ell\|^2_{\tilde{L}_2^{(m)*}(0,1]} = (\ell, \psi_\ell). \tag{2.15}$$

Therefore, we solve equation (2.14). Integrating by parts the expression on the right hand-side of (2.14) and taking into account the periodicity of functions f and ψ_ℓ, we have

$$\int_0^1 \ell(x) f(x)\mathrm{d}x = \int_0^1 (-1)^m \bar{\psi}_\ell^{(2m)}(x) f(x)\mathrm{d}x.$$

This means that the extremal function ψ_ℓ is a solution of the equation

$$\bar{\psi}_\ell^{(2m)}(x) = (-1)^m \ell(x). \tag{2.16}$$

Hence we have the following.

Lemma 1. *For a periodic integrable function $p(x) = v(x) + \mathrm{i}w(x)$ the extremal function ψ_ℓ of the error functional ℓ, satisfying equation (2.16), is determined by the formula*

$$\psi_\ell(x) = (-1)^m \left(\int_0^1 \bar{p}(t) B_{2m}(x-t)\mathrm{d}t - \sum_{k=1}^N \bar{C}_k B_{2m}(x-hk) + d_0 \right), \tag{2.17}$$

where d_0 is a constant, $\bar{p}(x)$ and $\bar{C}_k$ are complex conjugates of $p(x)$ and C_k, respectively, and $B_{2m}(x)$ is the Bernoulli polynomial defined by (2.9).

The proof of Lemma 1 is similar to the proof of Lemma 1 of [26].

Now, from (2.15), using (2.3) and (2.17), and taking into account the periodicity of $p(x)$ and $B_{2m}(x)$, after some calculations we get

$$\|\ell\|^2_{\tilde{L}_2^{(m)*}(0,1]} = \int_0^1 \ell(x)\psi_\ell(x)\mathrm{d}x = (-1)^m \left(\sum_{k=1}^N \sum_{k'=1}^N C_k \bar{C}_{k'} B_{2m}(hk - hk') \right.$$
$$\left. - \sum_{k=1}^N \int_0^1 \left(\bar{C}_k p(x) + C_k \bar{p}(x)\right) B_{2m}(x-hk)\mathrm{d}x + \int_0^1 \int_0^1 p(x)\bar{p}(t) B_{2m}(x-t)\mathrm{d}x\mathrm{d}t \right).$$

Hence, keeping in mind that $C_k = V_k + \mathrm{i}W_k$ and $p(x) = v(x) + \mathrm{i}w(x)$, we have

$$\|\ell\|^2_{\tilde{L}_2^{(m)*}(0,1]} = (-1)^m \left(\sum_{k=1}^N \sum_{k'=1}^N (V_k V_{k'} + W_k W_{k'}) B_{2m}(hk - hk') \right.$$
$$\left. -2 \sum_{k=1}^N \int_0^1 (V_k v(x) + W_k w(x)) B_{2m}(x-hk)\mathrm{d}x + \int_0^1 \int_0^1 (v(x)v(t) + w(x)w(t)) B_{2m}(x-t)\mathrm{d}x\mathrm{d}t \right), \tag{2.18}$$

where based on equality (2.5), for V_k and W_k, $k = 1, 2, \ldots, N$, we get the following conditions:

$$\sum_{k=1}^N V_k = \int_0^1 v(x)\mathrm{d}x, \quad \sum_{k=1}^N W_k = \int_0^1 w(x)\mathrm{d}x. \tag{2.19}$$

2.2. Minimization of the norm of the error functional ℓ

Now we search the minimum of the expression (2.18) by V_k and W_k, $k = 1, 2, \ldots, N$, under the conditions (2.19). As usual for this we consider the Lagrange function

$$\Lambda(\mathbf{V}, \mathbf{W}, \lambda_1, \lambda_2) = \|\ell\|^2_{\tilde{L}_2^{(m)*}(0,1]} + 2(-1)^m \left(\lambda_1 \left(\sum_{k=1}^N V_k - \int_0^1 v(x)\mathrm{d}x \right) + \lambda_2 \left(\sum_{k=1}^N W_k - \int_0^1 w(x)\mathrm{d}x \right) \right),$$

where $\mathbf{V} = (V_1, \ldots, V_N)$ and $\mathbf{W} = (W_1, \ldots, W_N)$.

Equating partial derivatives of the function $\Lambda(\mathbf{V}, \mathbf{W}, \lambda_1, \lambda_2)$ by V_k, W_k, $k = 1, 2, \ldots, N$, λ_1 and λ_2 to zero, we get the following two separate systems of linear equations for V_k, λ_1 and W_k, λ_2, respectively:

$$\sum_{k=1}^N V_k B_{2m}(hk' - hk) + \lambda_1 = \int_0^1 v(x) B_{2m}(x - hk')\mathrm{d}x, \;\; k' = 1, \ldots, N, \tag{2.20}$$

$$\sum_{k=1}^N V_k = \int_0^1 v(x)\mathrm{d}x \tag{2.21}$$

and

$$\sum_{k=1}^N W_k B_{2m}(hk' - hk) + \lambda_2 = \int_0^1 w(x) B_{2m}(x - hk')\mathrm{d}x, \;\; k' = 1, \ldots, N, \tag{2.22}$$

$$\sum_{k=1}^N W_k = \int_0^1 w(x)\mathrm{d}x. \tag{2.23}$$

The existence and uniqueness of solution of the systems (2.20)–(2.21) and (2.22)–(2.23) are proved similarly to the existence and uniqueness of solution of the system (16)–(17) from the work [26].

Hence, multiplying both sides of equalities (2.22) and (2.23) by i and adding to both sides of (2.20) and (2.21), respectively, we get the following system for $C_k = V_k + \mathrm{i}W_k$, $k = 1, 2, \ldots, N$ and $\lambda = \lambda_1 + \mathrm{i}\lambda_2$:

$$\sum_{k=1}^{N} C_k B_{2m}(hk' - hk) + \lambda = \int_0^1 p(x) B_{2m}(x - hk')\mathrm{d}x, \ \ k' = 1, \ldots, N, \tag{2.24}$$

$$\sum_{k=1}^{N} C_k = \int_0^1 p(x)\mathrm{d}x. \tag{2.25}$$

The solution $\mathring{C}_k$, $k = 1, 2, \ldots, N$, and $\mathring{\lambda}$ of the system (2.24)–(2.25) are optimal coefficients of the quadrature formula (2.1) and the corresponding Lagrange multiple.

2.3. *Coefficients of the optimal quadrature formula of the form* (2.1)

Further, we solve the system (2.24)–(2.25). Therefore we rewrite the system (2.24)–(2.25) in the convolution form as follows:

$$B_{2m}(hk) * (C_k \chi_{(0,1]}(hk)) + \lambda = \int_0^1 p(x) B_{2m}(x - hk)\mathrm{d}x, \ k = 1, \ldots, N, \tag{2.26}$$

$$\sum_{k=1}^{N} C_k = \int_0^1 p(x)\mathrm{d}x. \tag{2.27}$$

Applying the operator $hD_m(hk)*$ to both sides of Eq. (2.26) we have

$$\begin{aligned} &hD_m(hk) * \big(B_{2m}(hk) * (C_k \ \chi_{(0,1]}(hk)) + \lambda\big) \\ &\qquad = hD_m(hk) * \int_0^1 p(x) B_{2m}(x - hk)\mathrm{d}x, \quad k = 1, \ldots, N. \end{aligned} \tag{2.28}$$

Using the equalities (2.7), (2.8) and (2.10), for optimal coefficients from (2.28) we get the following:

$$C_k - h\sum_{k=1}^{N} C_k = hD_m(hk) * \int_0^1 p(x) B_{2m}(x - hk)\mathrm{d}x.$$

Hence, taking (2.27) into account, we get (2.11) which is the analytic formula for coefficients of the optimal quadrature formula (2.1) in the space $\widetilde{L}_2^{(m)*}(0, 1]$ as it was stated in Theorem 1.

2.4. *Optimal quadrature formulas for Fourier coefficients for the interval* $(0, 1]$

Further we consider the case $p(x) = \mathrm{e}^{2\pi \mathrm{i}\omega x}$ with $\omega \in \mathbb{Z}\backslash\{0\}$, then from (2.11), using (2.9), we have

$$\begin{aligned} \mathring{C}_k &= h\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\left(1 + (-1)^m \sum_{\gamma \neq 0} \frac{\mathrm{e}^{-2\pi \mathrm{i}\gamma x}}{(2\pi\gamma)^{2m}} \, D_m(hk) * \mathrm{e}^{2\pi \mathrm{i}\gamma hk}\right)\mathrm{d}x \\ &= h\left(\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\mathrm{d}x + (-1)^m \sum_{\gamma \neq 0} \frac{1}{(2\pi\gamma)^{2m}} \int_0^1 \mathrm{e}^{2\pi \mathrm{i}(\omega-\gamma)x}\mathrm{d}x \, D_m(hk) * \mathrm{e}^{2\pi \mathrm{i}\gamma hk}\right) \end{aligned}$$

Hence, keeping in mind the equalities

$$\int_0^1 \mathrm{e}^{2\pi \mathrm{i}\omega x}\mathrm{d}x = 0, \quad \int_0^1 \mathrm{e}^{2\pi \mathrm{i}(\omega-\gamma)x}\mathrm{d}x = \begin{cases} 1, & \omega = \gamma, \\ 0, & \omega \neq \gamma \end{cases}$$

for $\omega \in \mathbb{Z}\backslash\{0\}$, we have

$$\mathring{C}_k = \frac{(-1)^m h}{(2\pi\omega)^{2m}} \, D_m(hk) * \mathrm{e}^{2\pi \mathrm{i}\omega hk}, \quad k = 1, 2, \ldots, N. \tag{2.29}$$

Now we calculate the convolution in (2.29)

$$D_m(hk) * \mathrm{e}^{2\pi \mathrm{i}\omega hk} = \sum_{\gamma=-\infty}^{\infty} D_m(h\gamma) \mathrm{e}^{2\pi \mathrm{i}\omega(hk - h\gamma)}.$$

Hence by direct calculations of the last expression and putting the result to (2.29), we get (2.12) which was presented in Corollary 1.

3. An approximation formula for Fourier integrals in the interval [a,b]

Here we obtain an approximation formula for numerical computation of the integral (1.1) with $\omega \in \mathbb{R}$ for functions of the space $L_2^{(m)}[a,b]$.

In the previous section for the integral $\int_0^1 e^{2\pi i\omega x}f(x)dx$ with $\omega \in \mathbb{Z}$ in the space $\tilde{L}_2^{(m)}(0,1]$ we have obtained the optimal quadrature formula (OQF) of the form (2.1) with the weight function $p(x) = e^{2\pi i\omega x}$, and with coefficients (2.12) and (2.13). One of the expansions of these optimal quadrature formulas for the case $\omega \in \mathbb{R}$ is an approximation formula which is obtained by supposing the coefficients (2.12) as continuous functions with respect to real ω. Then for a function f of the space $L_2^{(m)}[a,b]$, using Corollaries 1 and 2, we get the following approximation formula:

$$\int_a^b e^{2\pi i\omega x}f(x)dx \cong \sum_{k=0}^{N} C_{k,\omega}[a,b]f(x_k) \tag{3.1}$$

where

$$\begin{aligned} C_{0,\omega}[a,b] &= \frac{h}{2}K_{\omega,m}e^{2\pi i\omega a},\\ C_{k,\omega}[a,b] &= hK_{\omega,m}e^{2\pi i\omega(hk+a)},\ \ k=1,2,\ldots,N-1,\\ C_{N,\omega}[a,b] &= \frac{h}{2}K_{\omega,m}e^{2\pi i\omega b} \end{aligned} \tag{3.2}$$

are coefficients, $x_k = hk + a$, $k = 0,1,\ldots,N$, are nodes of the formula (3.1), $\omega \in \mathbb{R}$, $i^2 = -1$, $h = (b-a)/N$ with $N \in \mathbb{N}$ and

$$K_{\omega,m} = \begin{cases} \left(\frac{\sin \pi\omega h}{\pi\omega h}\right)^{2m} \dfrac{(2m-1)!}{2\sum\limits_{l=0}^{m-2} a_l \cos[2\pi\omega h(m-1-l)] + a_{m-1}} & \text{for } \omega \in \mathbb{R}\backslash\{0\},\\ 1 & \text{for } \omega = 0, \end{cases} \tag{3.3}$$

where a_l are the coefficients of the Euler–Frobenius polynomial of $(2m-2)$ degree.

Remark 1. It should be noted that for $\omega = 0$ from approximation formula (3.1) with coefficients (3.2) we get the trapezoidal quadrature formula which is optimal in the Sobolev space $L_2^{(1)}[a,b]$ of non-periodic functions and in the Sobolev space $\tilde{L}_2^{(m)}[a,b]$ of periodic functions [17,18,29].

4. Application of the approximation formulas for reconstruction of CT images

Algorithm 1 Pseudo code for CT image reconstruction using the approximation formula (3.1).

1: A given projection data (sinogram):

$$P(t,\theta) = \int_{\ell_{t,\theta}} f(x,y)ds, \quad 0 \le \theta \le \pi, \quad a \le t \le b.$$

2: Numerical calculation of the Fourier transform $S(\omega,\theta)$ of $P(t,\theta)$ using the approximation formula (3.1):

$$S(\omega,\theta) \cong S(\omega,\theta_k) = \sum_{m=0}^{M} C_{m,-\omega}P(t_m,\theta_k), \quad \omega \in \mathbb{R}.$$

3: Numerical computation of the inverse Fourier transform $Q(t,\theta)$ of $S(\omega,\theta)|\omega|$ using the approximation formula (3.1):

$$Q(t,\theta) \cong Q(t,\theta_k) = \sum_{n=0}^{N} C_{n,t}S(\omega_n,\theta_k)|\omega_n|, \quad t \in \mathbb{R}.$$

4: Backprojection to reconstruct the CT image:

$$f(x,y) = \int_0^{\pi} Q(t,\theta)d\theta \cong \frac{\pi}{K}\sum_{k=0}^{K-1} Q(t,\theta_k).$$

In this section we apply the approximation formula (3.1) with coefficients (3.2) for reconstruction of CT images based on the filtered back-projection method.

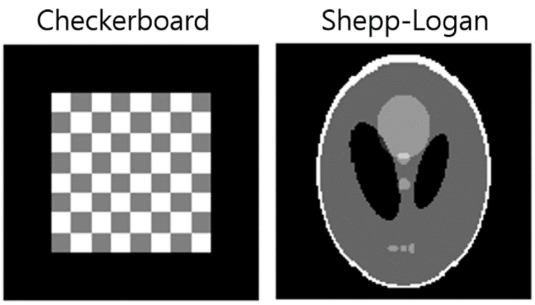

Fig. 1. Original phantom images of size 128 × 128: checkerboard (left) and the Shepp–Logan (right).

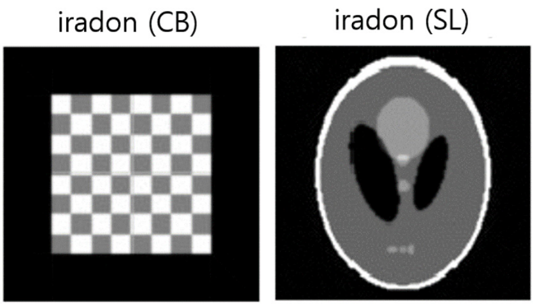

Fig. 2. Reconstructed CT image using the MATLAB built-in function *iradon*: checkerboard (left) and the Shepp–Logan (right).

The problem of CT is to reconstruct the function $\mu(x,y)$ from its Radon projections

$$P(t,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)\delta(x\cos\theta+y\sin\theta-t)\mathrm{d}x\mathrm{d}y,$$

where δ denotes the Dirac delta-function. The simplest projection $P(t,\theta)$ is a collection of parallel ray integrals along a line $\ell_{t,\theta}$: $x\cos\theta+y\sin\theta=t$ perpendicular to a unit vector $(\cos\theta,\sin\theta)$ with the distance t to the origin for a constant θ. This is known as a parallel beam projection. There are fan-beam in 2D and cone-beam in 3D projections [1–3].

It should be noted that we have analytic and iterative methods for CT reconstruction. One of the widely used analytic methods of CT reconstruction, as was mentioned in the Introduction, is the filtered back-projection method. It can be modeled by

$$\mu(x,y)=\int_0^{\pi}\int_{-\infty}^{\infty}S(\omega,\theta)\,|\omega|\,\mathrm{e}^{2\pi\mathrm{i}\omega(x\cos\theta+y\sin\theta)}\mathrm{d}\omega\mathrm{d}\theta, \tag{4.1}$$

where

$$S(\omega,\theta)=\int_{-\infty}^{\infty}P(t,\theta)\mathrm{e}^{-2\pi\mathrm{i}\omega t}\mathrm{d}t \tag{4.2}$$

is the 1D Fourier transform of $P(t,\theta)$. The inner integral of (4.1) can be regarded as a 1D inverse Fourier transform of the product $S(\omega,\theta)\,|\omega|$, i.e.,

$$Q(t,\theta)=\int_{-\infty}^{\infty}S(\omega,\theta)|\omega|\mathrm{e}^{2\pi\mathrm{i}\omega t}\mathrm{d}\omega, \tag{4.3}$$

which represents a projection filtered by a 1D filter whose frequency representation is $|\omega|$. The outer integral performs back-projection. Therefore, the filtered back-projection consists of two steps: filtration and then back-projection.

In practical implementation of the filtered back-projection method (4.1)–(4.3) we have finite discrete values of the Radon transform. Therefore, we can give an algorithm (see, Algorithm 1) for approximate reconstruction of CT images based on the approximation formula (3.1) with coefficients (3.2). As we can see in the pseudo code of Algorithm 1, the approximation formula (3.1) is applied in Steps 2 and 3 for numerical calculation of the Fourier transforms.

Two simulated phantoms, a checkerboard (CB) and the Shepp–Logan (SL), are used for the numerical experiment (Fig. 1). To generate the sinograms, half rotation sampling with sampling angle 1° is adopted for the phantom image of size 128 × 128. For the comparison of performance of the proposed algorithm, a built-in function *iradon* of MATLAB (version: MATLAB 2019a) is used and the reconstructed image of *iradon* is given in Fig. 2.

Table 1
Quantitative image analysis for the CT image reconstruction for Checkerboard image. All metrics are compared with those of *iradon*.

Checkerboard	iradon	$m=1$	$m=2$	$m=3$
$E_{\max}$	0.3058	0.3615	0.2965	0.2832
MSE	0.0022	0.0044	0.0017	0.0016
PSNR	26.6692	23.6016	27.6976	28.0568

Table 2
Quantitative image analysis for the CT image reconstruction for the Shepp–Logan phantom. All metrics are compared with those of *iradon*.

Shepp–Logan	iradon	$m=1$	$m=2$	$m=3$
$E_{\max}$	0.3601	0.3960	0.3357	0.3307
MSE	0.0036	0.0042	0.0028	0.0026
PSNR	24.4305	23.7804	25.5892	25.8492

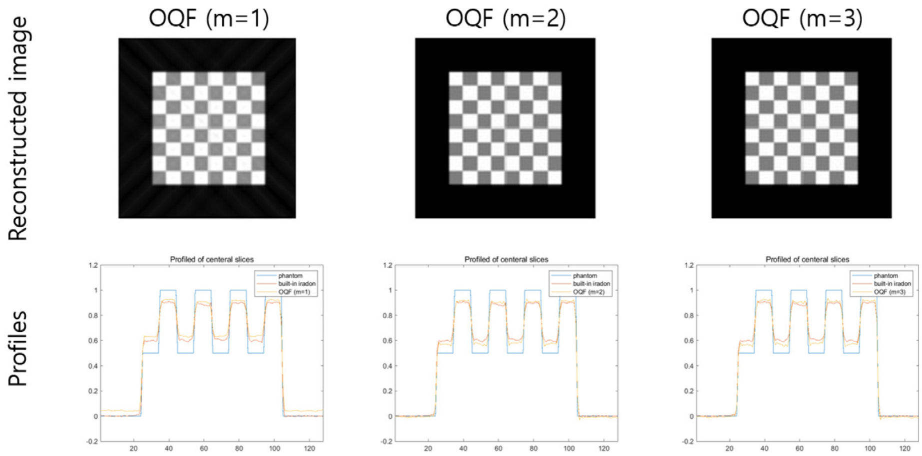

Fig. 3. Results of the proposed algorithm using the approximation formula (3.1) of the first order (first column), of the second order (second column), and of the third order (third column) for the checkerboard: the reconstructed images (top) and the profile lines for comparison (bottom).

For the image quality analysis, we compare maximum error ($E_{\max}$), mean squared error (MSE), and the peak signal-to-noise ratio (PSNR):

$$E_{\max}(I) = \max_{i,j} |I(i,j) - I_{ref}(i,j)|,$$

$$\mathrm{MSE}(I) = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} |I(i,j) - I_{ref}(i,j)|^2,$$

$$\mathrm{PSNR}(I) = 10 \log_{10} \left(\frac{I_{\max}^2}{\mathrm{MSE}(I)} \right),$$

where $I_{\max}$ is the maximum pixel value of the image I. Images of original simulated phantoms are used for I_{ref} (Fig. 1).

Figs. 3 and 4 show the results of CT image reconstruction using the proposed algorithm for a checkerboard image and Shepp–Logan phantom, respectively. Compared with Fig. 2, the proposed algorithm also produces the clean reconstructed images for the approximation formulas of $m = 1$, 2, and 3. As shown in Figs. 3 and 4, the approximation formulas produce the clean reconstructed images, which have the same structures with the results of *iradon* (Fig. 2). Tables 1 and 2 show $E_{\max}$, MSE, and PSNR for the reconstruction results of approximation algorithms for a checkerboard image and the Shepp–Logan phantom, respectively. The higher the value of m, the better the image quality and for the cases $m \geq 2$, all metrics are better than those of *iradon* results for both phantom.

In [24], the optimal quadrature formulas were constructed for numerical integration of the Fourier integral (1.1) in the Sobolev space $L_2^{(m)}[a, b]$ of complex-valued non-periodic functions. The optimal quadrature formulas of the cases $m = 2$ and $m = 3$ were applied for reconstruction of Computed Tomography images. Compared with the optimal quadrature formulas in non-periodic case, the approximation formula (3.1) with coefficients (3.2) for the periodic case is much simpler, therefore it is easy to implement and costs less computation even though both provide the similar performances (see Section 3 of [24]).

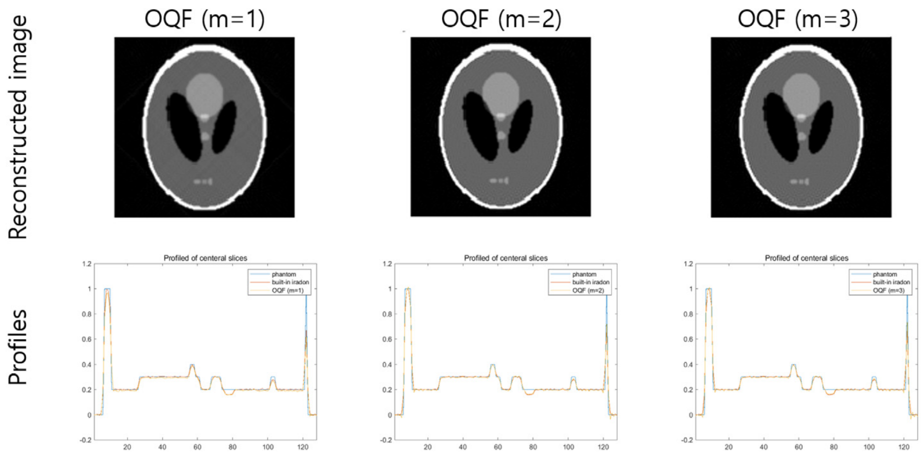

Fig. 4. Results of the proposed algorithm using the approximation formula (3.1) of the first order (first column), of the second order (second column), and of the third order (third column) for the Shepp–Logan phantom: the reconstructed images (top) and the profile lines for comparison (bottom).

Acknowledgments

The work has been done while A.R. Hayotov was visiting Department of Mathematical Sciences at KAIST, Daejeon, Republic of Korea. A.R. Hayotov's work was supported by the 'Korea Foundation for Advanced Studies'/'Chey Institute for Advanced Studies' International Scholar Exchange Fellowship for academic year of 2018–2019. We are very thankful to professor C.-O. Lee for discussion of the results of this work and permanent attention.

We are very grateful to the reviewer for remarks and suggestions, which have improved the quality of the paper.

References

[1] T.M. Buzug, Computed Tomography from Photon Statistics to Modern Cone-Beam CT, Springer, Berlin, 2008.
[2] T.G. Feeman, The Mathematics of Medical Imaging, A Beginner's Guide, second ed., Springer, 2015.
[3] A.C. Kak, M. Slaney, Principles of Computerized Tomographic Imaging, IEEE Press, New York, 1988.
[4] L.N.G. Filon, On a quadrature formula for trigonometric integrals, Proc. Roy. Soc. Edinburgh 49 (1928) 38–47.
[5] V.A. Avdeenko, A.A. Malyukov, A quadrature formula for the fourier integral based on the use of a cubic spline, USSR Comput. Math. Math. Phys. 29 (1989) 783–786, (Russian).
[6] I. Babuška, Optimal quadrature formulas, Dokl. Akad. Nauk. SSSR 149 (1963) 227–229, (Russian).
[7] I. Babuška, E. Vitasek, M. Prager, Numerical Processes in Differential Equations, Wiley, New York, 1966.
[8] N.S. Bakhvalov, L.G. Vasil'eva, Evaluation of the integrals of oscillating functions by interpolation at nodes of Gaussian quadratures, USSR Comput. Math. Math. Phys. 8 (1968) 241–249, (Russian).
[9] A. Iserles, S.P. Nørsett, Efficient quadrature of highly oscillatory integrals using derivatives, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005) 1383–1399.
[10] G.V. Milovanović, Numerical calculation of integrals involving oscillatory and singular kernels and some applications of quadratures, Comput. Math. Appl. 36 (1998) 19–39.
[11] E. Novak, M. Ullrich, H. Woźniakowski, Complexity of oscillatory integration for univariate Sobolev space, J. Complexity 31 (2015) 15–41.
[12] Z. Xu, G.V. Milovanović, S. Xiang, Efficient computation of highly oscillatory integrals with Henkel kernel, Appl. Math. Comput. 261 (2015) 312–322.
[13] S. Zhang, E. Novak, Optimal quadrature formulas for the Sobolev space H^1, J. Sci. Comput. 78 (2019) 274–289.
[14] A. Deaño, D. Huybrechs, A. Iserles, Computing Highly Oscillatory Integrals, SIAM, Philadelphia, 2018.
[15] G.V. Milovanović, M.P. Stanić, Numerical integration of highly oscillating functions, in: G.V. Milovanović, M.Th. Rassias (Eds.), Analytic Number Theory, Approximation Theory, and Special Functions/, Springer, New York, 2014, pp. 613–649.
[16] S. Olver, Numerical Approximation of Highly Oscillatory Integrals, (PhD dissertation), University of Cambridge, 2008.
[17] S.L. Sobolev, Introduction To the Theory of Cubature Formulas, Nauka, Moscow, 1974, (Russian).
[18] S.L. Sobolev, V.L. Vaskevich, The Theory of Cubature Formulas, Kluwer Academic Publishers Group, Dordrecht, 1997.
[19] J.H. Ahlberg, E.N. Nilson, J.L. Walsh, The Theory of Splines and their Applications, Mathematics in Science and Engineering, Academic Press, New York, 1967.
[20] N.D. Boltaev, A.R. Hayotov, Kh.M. Shadimetov, Construction of optimal quadrature formula for numerical calculation of Fourier coefficients in Sobolev space $L_2^{(1)}$, Am. J. Numer. Anal. 4 (2016) 1–7.
[21] N.D. Boltaev, A.R. Hayotov, Kh.M. Shadimetov, Construction of optimal quadrature formulas for Fourier coefficients in Sobolev space $L_2^{(m)}(01)$, Numer. Algorithms 74 (2017) 307–336.
[22] N.D. Boltaev, A.R. Hayotov, G.V. Milovanović, Kh.M. Shadimetov, Optimal quadrature formulas for Fourier coefficients in $W_2^{(mm-1)}$ space, J. Appl. Anal. Comput. 7 (2017) 1233–1266.
[23] A.R. Hayotov, S.M. Jeon, C.-O. Lee, On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$, J. Comput. Appl. Math. 372 (2020) 112713.
[24] A.R. Hayotov, S.M. Jeon, C.-O. Lee, Kh.M. Shadimetov, Optimal quadrature formulas for non-periodic functions in Sobolev space and its application to CT image reconstruction, 2020, arXiv:2001.02636v2 [math.NA].
[25] Kh.M. Shadimetov, On an optimal quadrature formula, Uzbek Math. Zh. (3) (1998) 90–98.

[26] Kh.M. Shadimetov, Weight optimal cubature formulas in Sobolev's periodic space, Siberian J. Numer. Math. -Novosibirsk 2 (1999) 185–196, (Russian).
[27] Kh.M. Shadimetov, The discrete analogue of the differential operator d^{2m}/dx^{2m} and its construction, in: Questions of Computations and Applied Mathematics. Tashkent, no. 79, 1985, pp. 22-35. ArXiv:1001.0556v1 [math.NA], 2010.
[28] F.G. Frobenius, Über die Bernoullischen Zahlen und Eulerischen Polynome. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1910) 809-847.
[29] S.L. Sobolev, The coefficients of optimal quadrature formulas, in: Selected Works of S.L. Sobolev, Springer, US, 2006, pp. 561–566.